Model Selection and Error Estimation

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Abstract

We study model selection strategies based on penalized empirical loss minimization. We point out a tight relationship between error estimation and data-based complexity penalization: any good error estimate may be converted into a data-based penalty function and the performance of the estimate is governed by the quality of the error estimate. We consider several penalty functions, involving error estimates on independent test data, empirical VC dimension, empirical VC entropy, and margin-based quantities. We also consider the maximal difference between the error on the first half of the training data and the second half, and the expected maximal discrepancy, a closely related capacity estimate that can be calculated by Monte Carlo integration. Maximal discrepancy penalty functions are appealing for pattern classification problems, since their computation is equivalent to empirical risk minimization over the training data with some labels flipped.

1 INTRODUCTION

We consider the following prediction problem. Based on a random observation $X \in \mathcal{X}$, one has to estimate $Y \in \mathcal{Y}$. A *prediction rule* is a measurable function $f : \mathcal{X} \to \mathcal{Y}$, with *loss* $L(f) = \mathbf{E}\ell(f(X), Y)$, where $\ell : \mathcal{Y} \times \mathcal{Y} \to [0, 1]$ is a bounded loss function. The data

$$D_n = (X_1, Y_1), \ldots, (X_n, Y_n)$$

consist of a sequence of independent, identically distributed samples with the same distribution as (X, Y) and D_n is independent of (X, Y). The goal is to choose a prediction rule f_n from some restricted class $\mathcal{F}$ such that the *loss* $L(f_n) = \mathbf{E}[\ell(f_n(X), Y)|D_n]$ is as close as possible to the best possible loss, $L^* = \inf_f L(f)$, where the infimum is taken over all prediction rules $f : \mathcal{X} \to \mathcal{Y}$.

Empirical risk minimization evaluates the performance of each prediction rule $f \in \mathcal{F}$ in terms of its empirical loss $\widehat{L}_n(f) = \frac{1}{n}\sum_{i=1}^n \ell(f(X_i), Y_i)$. This provides an estimate whose loss is close to the optimal loss L^* if the class $\mathcal{F}$ is (i) sufficiently large so that the loss of the best function in $\mathcal{F}$ is close to L^* and (ii) is sufficiently small so that finding the best candidate in $\mathcal{F}$ based on the data is still possible. These two requirements are clearly in conflict. The trade-off is best understood by writing

$$\begin{aligned}&\mathbf{E}L(f_n) - L^* \\ &\quad = \left(\mathbf{E}L(f_n) - \inf_{f\in\mathcal{F}} L(f)\right) + \left(\inf_{f\in\mathcal{F}} L(f) - L^*\right).\end{aligned}$$

The first term is often called *estimation error*, while the second is the *approximation error*. Often $\mathcal{F}$ is large enough to minimize $L(\cdot)$ for all possible distributions of (X, Y), so that $\mathcal{F}$ is too large for empirical risk minimization. In this case it is common to fix in advance a sequence of smaller model classes $\mathcal{F}_1, \mathcal{F}_2, \ldots$ whose union is equal to $\mathcal{F}$. Given the data D_n, one wishes to select a good model from *one* of these classes. This is the problem of model selection.

Denote by $\widehat{f}_k$ a function in $\mathcal{F}_k$ having minimal empirical risk. One hopes to select a model class $\mathcal{F}_K$ such that the excess error $\mathbf{E}L(\widehat{f}_K) - L^*$ is close to

$$\begin{aligned}&\min_k \mathbf{E}L(\widehat{f}_k) - L^* = \\ &\quad \min_k \left[\left(\mathbf{E}L(\widehat{f}_k) - \inf_{f\in\mathcal{F}_k} L(f)\right) + \left(\inf_{f\in\mathcal{F}_k} L(f) - L^*\right)\right].\end{aligned}$$

The idea of *structural risk minimization*, (also known as *complexity regularization*, is to add a complexity penalty to each of the $\widehat{L}_n(\widehat{f}_k)$'s to compensate for the overfitting effect. This penalty is usually closely related to a distribution-free upper bound for $\sup_{f\in\mathcal{F}_k} |\widehat{L}_n(f) - L(f)|$ so that the penalty eliminates the effect of overfitting. Thus, structural risk minimization finds the best trade-off between the approximation error and a distribution-free upper bound on the estimation error. Unfortunately, distribution-free upper bounds may be too conservative for specific distributions. This criticism has led to the idea of using *data-dependent* penalties.

In the next section, we show that any approximate upper bound on error (including a data-dependent bound) can be used to define a (possibly data-dependent) complexity penalty $C_n(k)$ and a model selection algorithm for which the excess error is close to

$$\min_k \left[\mathbf{E}C_n(k) + \left(\inf_{f\in\mathcal{F}_k} L(f) - L^*\right)\right].$$

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Section 3 reviews some concentration inequalities that are central to our proofs. Section 4 gives several applications of the performance bounds of Section 2: Section 4.1 considers the estimates provided by an independent test sample. These have the disadvantage that they cost data. Section 4.2, considers a distribution-free estimate based on the VC dimension and a data-dependent estimate based on shatter coefficients. Unfortunately, these are difficult to compute. Section 4.3 briefly considers margin-based error estimates, which can be viewed as easily computed estimates of quantities analogous to shatter coefficients. Section 4.4 looks at an estimate provided by maximizing the discrepancy between the error on the first half of the sample and that on the second half. For classification, this estimate can be conveniently computed, simply by minimizing empirical risk with half of the labels flipped. Section 4.5 looks at a more complex estimate: the expected maximum discrepancy. This estimate can be calculated by Monte Carlo integration, and can lead to better performance bounds.

For clarity, we include in Table 1 notation that we use throughout the paper.

For work on complexity regularization, see [1, 2, 3, 4, 5, 8, 9, 11, 12, 14, 16, 17, 20, 21, 22, 24, 23, 25, 29, 30, 31, 33, 34, 35, 38, 42, 46, 47]. Data-dependent penalties are studied in [22, 6, 15, 34].

2 PENALIZATION BY ERROR ESTIMATES

For each class $\mathcal{F}_k$, let $\widehat{f}_k$ denote the prediction rule that is selected from $\mathcal{F}_k$ based on the data. Our goal is to select, among these rules, one which has approximately minimal loss. The key assumption for our analysis is that the true loss of $\widehat{f}_k$ can be estimated for all k.

Assumption 1 *There are positive numbers c and m such that for each k an estimate $R_{n,k}$ on $L(\widehat{f}_k)$ is available which satisfies*

$$\mathbf{P}\left[L(\widehat{f}_k) > R_{n,k} + \epsilon\right] \le ce^{-2m\epsilon^2} \tag{1}$$

for all ϵ.

Now define the data-based complexity penalty by

$$C_n(k) = R_{n,k} - \widehat{L}_n(\widehat{f}_k) + \sqrt{\frac{\log k}{m}}.$$

The last term is required because of technical reasons that will become apparent shortly. It is typically small. The difference $R_{n,k} - \widehat{L}_n(\widehat{f}_k)$ is simply an estimate of the 'right' amount of penalization $L(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k)$. Finally, define the prediction rule:

$$f_n = \operatorname*{arg\,min}_{k=1,2,\ldots} \tilde{L}_n(\widehat{f}_k),$$

where

$$\tilde{L}_n(\widehat{f}_k) = \widehat{L}_n(\widehat{f}_k) + C_n(k) = R_{n,k} + \sqrt{\frac{\log k}{m}}.$$

The following theorem summarizes the main performance bound for f_n.

Theorem 1 *Assume that the error estimates $R_{n,k}$ satisfy (1) for some positive constants c and m. Then for all $\epsilon > 0$,*

$$\mathbf{P}\left[L(f_n) - \tilde{L}_n(f_n) > \epsilon\right] \le 2ce^{-2m\epsilon^2}.$$

Moreover, if for all k, $\widehat{f}_k$ minimizes the empirical loss in the model class $\mathcal{F}_k$, then

$$\begin{aligned}&\mathbf{E}L(f_n) - L^* \\ &\le \min_k \left[\mathbf{E}C_n(k) + \left(\inf_{f\in\mathcal{F}_k} L(f) - L^*\right)\right] + \sqrt{\frac{\log(ce)}{2m}}.\end{aligned}$$

The second part of Theorem 1 shows that the prediction rule minimizing the penalized empirical loss achieves an almost optimal trade-off between the approximation error and the expected complexity, provided that the estimate $R_{n,k}$ on which the complexity is based is an approximate upper bound on the loss. In particular, if we knew in advance which of the classes $\mathcal{F}_k$ contained the optimal prediction rule, we could use the error estimates $R_{n,k}$ to obtain an upper bound on $\mathbf{E}L(\widehat{f}_k) - L^*$, and this upper bound would not improve on the bound of Theorem 1 by more than $O\left(\sqrt{\log k/m}\right)$.

If the range of the loss function ℓ is an infinite set, the infimum of the empirical loss might not be achieved. In this case, we could define $\widehat{f}_k$ as a suitably good approximation to the infimum. However, for convenience, we assume throughout that the minimum always exists. It suffices for this, and for various proofs, to assume that for all n and $(x_1, y_1), \ldots, (x_n, y_n)$, the set

$$\{(\ell(f(x_1), y_1), \ldots, \ell(f(x_1), y_1)) : f \in \mathcal{F}_k\}$$

is closed. This is certainly satisfied for pattern classification.

Proof. For brevity, introduce the notation

$$L_k^* = \inf_{f\in\mathcal{F}_k} L(f).$$

Then for any $\epsilon > 0$,

$$\begin{aligned}
&\mathbf{P}\left[L(f_n) - \tilde{L}_n(f_n) > \epsilon\right] \\
&\le \mathbf{P}\left[\sup_{j=1,2,\ldots}\left(L(\widehat{f}_j) - \tilde{L}_n(\widehat{f}_j)\right) > \epsilon\right] \\
&\le \sum_{j=1}^{\infty} \mathbf{P}\left[L(\widehat{f}_j) - \tilde{L}_n(\widehat{f}_j) > \epsilon\right] \\
&\quad \text{(by the union bound)} \\
&= \sum_{j=1}^{\infty} \mathbf{P}\left[L(\widehat{f}_j) - R_{n,j} > \epsilon + \sqrt{\frac{\log j}{m}}\right] \\
&\quad \text{(by definition)} \\
&\le \sum_{j=1}^{\infty} ce^{-2m\left(\epsilon + \sqrt{\frac{\log j}{m}}\right)^2} \quad \text{(by Assumption 1)} \\
&\le \sum_{j=1}^{\infty} ce^{-2m\left(\epsilon^2 + \frac{\log j}{m}\right)} \\
&< 2ce^{-2m\epsilon^2} \quad (\text{since } \textstyle\sum_{j=1}^{\infty} j^{-2} < 2).
\end{aligned}$$

f	prediction rule, $f : \mathcal{X} \to \mathcal{Y}$
$\mathcal{F}_1, \mathcal{F}_2, \ldots$	sets of prediction rules (model classes)
$\mathcal{F}$	union of model classes $\mathcal{F}_k$
f_k^*	element of F_k with minimal loss
$\widehat{f}_k$	element of $\mathcal{F}_k$ minimizing empirical loss
f_n	prediction rule from $\mathcal{F}$ minimizing $\tilde{L}_n(\widehat{f}_k)$
ℓ	loss function, $\ell : \mathcal{Y} \times \mathcal{Y} \to [0,1]$
L	loss, $L(f) = \mathbf{E}\ell(f(X), Y)$
L_k^*	minimal loss of functions in $\mathcal{F}_k$, $L_k^* = \inf_{f \in \mathcal{F}_k} L(f)$
$\widehat{L}_n$	empirical loss
$R_{n,k}$	estimate (high confidence upper bound) of loss $L(\widehat{f}_k)$
$C_n(k)$	complexity penalty for class $\mathcal{F}_k$
$\tilde{L}_n$	complexity penalized loss estimate, $\tilde{L}_n(\widehat{f}_k) = \widehat{L}_n(\widehat{f}_k) + C_n(k)$
L^*	loss of optimal prediction rule

Table 1: Notation.

To prove the second inequality, for each k, we decompose $L(f_n) - L_k^*$ as

$$\begin{aligned} & L(f_n) - L_k^* \\ &= \left(L(f_n) - \inf_j \tilde{L}_n(\widehat{f}_j) \right) + \left(\inf_j \tilde{L}_n(\widehat{f}_j) - L_k^* \right). \end{aligned}$$

The first term may be bounded, by standard integration of the tail inequality shown above (see, e.g., [14, page 208]), as $\mathbf{E}\left[L(f_n) - \inf_j \tilde{L}_n(\widehat{f}_j) \right] \le \sqrt{\log(ce)/(2m)}$. Choosing f_k^* such that $L(f_k^*) = L_k^*$, the second term may be bounded directly by

$$\begin{aligned} & \mathbf{E} \inf_j \tilde{L}_n(\widehat{f}_j) - L_k^* \\ & \le \mathbf{E}\tilde{L}_n(\widehat{f}_k) - L_k^* \\ & = \mathbf{E}\widehat{L}_n(\widehat{f}_k) - L_k^* + \mathbf{E}C_n(k) \\ & \quad \text{(by the definition of } \tilde{L}_n(\widehat{f}_k)) \\ & \le \mathbf{E}\widehat{L}_n(f_k^*) - L(f_k^*) + \mathbf{E}C_n(k) \\ & \quad \text{(since } \widehat{f}_k \text{ minimizes the empirical loss on } \mathcal{F}_k) \\ & = \mathbf{E}C_n(k), \end{aligned}$$

where the last step follows from the fact that $\mathbf{E}\widehat{L}_n(f_k^*) = L(f_k^*)$. Summing the obtained bounds for both terms yields that for each k,

$$\mathbf{E}L(f_n) \le \mathbf{E}C_n(k) + L_k^* + \sqrt{\log(ce)/(2m)},$$

which implies the second statement of the theorem. ∎

Sometimes bounds tighter than Assumption 1 are available, as in Assumption 2 below. Such bounds may be exploited to decrease the term $\sqrt{\log k/m}$ in the definition of the complexity penalty.

Assumption 2 *There are positive numbers c and m such that for each k an estimate $\overline{R}_{n,k}$ of $L(\widehat{f}_k)$ is available which satisfies*

$$\mathbf{P}\left[L(\widehat{f}_k) > \overline{R}_{n,k} + \epsilon \right] \le ce^{-m\epsilon} \qquad (2)$$

for all ϵ.

Define the modified penalty by

$$\overline{C}_n(k) = \overline{R}_{n,k} - \widehat{L}_n(\widehat{f}_k) + \frac{2\log k}{m}$$

and define the prediction rule

$$\overline{f}_n = \arg\min_{k=1,2,\ldots} \overline{L}_n(\widehat{f}_k),$$

where

$$\overline{L}_n(\widehat{f}_k) = \widehat{L}_n(\widehat{f}_k) + \overline{C}_n(k) = \overline{R}_{n,k} + \frac{2\log k}{m}.$$

Then by a trivial modification of the proof of Theorem 1 we obtain the following result.

Theorem 2 *Assume that the error estimates $\overline{R}_{n,k}$ satisfy Assumption 2 for some positive constants c and m. Then for all $\epsilon > 0$,*

$$\mathbf{P}\left[L(f_n) - \overline{L}_n(f_n) > \epsilon \right] \le 2ce^{-m\epsilon}.$$

Moreover, if for all k, $\widehat{f}_k$ minimizes the empirical loss in the model class $\mathcal{F}_k$, then

$$\begin{aligned} & \mathbf{E}L(\overline{f}_n) - L^* \\ & \le \min_k \left[\mathbf{E}\overline{C}_n(k) + \left(\inf_{f \in \mathcal{F}_k} L(f) - L^* \right) \right] + \frac{\log(2ec)}{m}. \end{aligned}$$

3 CONCENTRATION INEQUALITIES

Concentration-of-measure results are central to our analysis. These inequalities guarantee that certain functions of independent random variables are close to their mean. Here we recall three such inequalities.

Theorem 3 (McDiarmid [28]). *Let $X_1, \ldots, X_n$ be independent random variables taking values in a set A, and assume that $f : A^n \to R$ satisfies*

$$\sup_{\substack{x_1,\ldots,x_n,\\ x_i' \in A}} \Big| f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \Big| \leq c_i$$

for $1 \leq i \leq n$. Then for all $t > 0$

$$\mathbf{P}\left\{ f(X_1, \ldots, X_n) \geq \mathbf{E} f(X_1, \ldots, X_n) + t \right\} \leq e^{-2t^2 / \sum_{i=1}^n c_i^2}$$

and

$$\mathbf{P}\left\{ f(X_1, \ldots, X_n) \leq \mathbf{E} f(X_1, \ldots, X_n) - t \right\} \leq e^{-2t^2 / \sum_{i=1}^n c_i^2} .$$

McDiarmid's inequality is convenient when $f()$ has variance $\Theta(\sum_{i=1}^n c_i^2)$. In other situations when the variance of f is much smaller, the following inequality might be more appropriate.

Theorem 4 (Boucheron, Lugosi, and Massart [10]) *Suppose that $X_1, \ldots, X_n$ are independent random variables taking values in a set A, and that $f : A^n \to R$ is such that there exists a function $g : A^{n-1} \to R$ such that for all $x_1, \ldots, x_n \in A$*

(1) $f(x_1, \ldots, x_n) \geq 0$;
(2) $0 \leq f(x_1, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \leq 1$ *for all* $i = 1, \ldots, n$;
(3) $\sum_{i=1}^n [f(x_1, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)] \leq f(x_1, \ldots, x_n)$.

Then for any $t > 0$,

$$\mathbf{P}\left[f(X_1, \ldots, X_n) \geq \mathbf{E} f(X_1, \ldots, X_n) + t \right] \leq \exp\left[-\frac{t^2}{2\mathbf{E} f(X_1, \ldots, X_n) + 2t/3} \right],$$

and

$$\mathbf{P}\left[f(X_1, \ldots, X_n) \leq \mathbf{E} f(X_1, \ldots, X_n) - t \right] \leq \exp\left[-\frac{t^2}{2\mathbf{E} f(X_1, \ldots, X_n)} \right],$$

moreover,

$$\mathbf{E} f(X_1, \ldots, X_n) \leq \log_2 \mathbf{E}\left[2^{f(X_1, \ldots, X_n)} \right] \leq \frac{1}{\log 2} \mathbf{E} f(X_1, \ldots, X_n).$$

Finally, we recall a concentration inequality of van der Vaart and Wellner [44], obtained from one of Talagrand's isoperimetric inequalities [37].

Theorem 5 (van der Vaart and Wellner [44]). *Let A be a set, and let $f_n : A^n \to [0, n]$ be a permutation symmetric function satisfying the monotonicity and subadditive properties*

$$f_n(x) \leq f_{n+m}(x, y)$$

and

$$f_{n+m}(x, y) \leq f_n(x) + f_m(y)$$

for all $x \in A^n$ and $y \in A^m$. Then if $X_1, \ldots, X_n$ are i.i.d. random variables taking values in A, then for any $t > 0$,

$$\mathbf{P}[f_n(X_1, \ldots, X_n) > t] \leq \exp\left(-\frac{t}{2} \log\left(\frac{t}{8\mathbf{E} f_n + 4} \right) \right) .$$

4 APPLICATIONS

4.1 INDEPENDENT TEST SAMPLE

Assume that m independent sample pairs

$$(X_1', Y_1'), \ldots, (X_m', Y_m')$$

are available. We can simply remove m samples from the training data. Of course, this is not very attractive, but m may be small relative to n. In this case we can estimate $L(\widehat{f}_k)$ by

$$R_{n,k} = \frac{1}{m} \sum_{i=1}^m \ell(\widehat{f}_k(X_i'), Y_i'). \tag{3}$$

We apply Hoeffding's inequality to show that Assumption 1 is satisfied with $c = 1$, notice that $\mathbf{E}[R_{n,k} | D_n] = L(\widehat{f}_k)$, and apply Theorem 1 to give the following result.

Corollary 1 *Assume that the model selection algorithm of Section 2 is performed with the hold-out error estimate (3). Then*

$$\mathbf{E} L(f_n) - L^* \leq \min_k \left[\mathbf{E}\left[L(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k) \right] + \left(\inf_{f \in \mathcal{F}_k} L(f) - L^* \right) + \sqrt{\frac{\log k}{m}} \right] + \frac{1}{\sqrt{2m}} .$$

In other words, the estimate achieves a nearly optimal balance between the approximation error, and the quantity

$$\mathbf{E}\left[L(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k) \right],$$

which may be regarded as the amount of overfitting.

With this inequality we recover the main result of Lugosi and Nobel [22], but now with a much simpler estimate. In fact, the bound of the corollary may substantially improve the main result of [22].

The square roots in the bound of Corollary 1 can be removed by increasing the penalty term by a small constant factor and using Bernstein's inequality in place of Hoeffding's. We omit the details.

4.2 ESTIMATED COMPLEXITY

In the remaining examples we consider error estimates $R_{n,k}$ which avoid splitting the data.

For simplicity, we concentrate in this section on the case of classification ($\mathcal{Y} = \{0,1\}$ and the *0-1 loss*, defined by $\ell(0,0) = \ell(1,1) = 0$ and $\ell(0,1) = \ell(1,0) = 1$), although similar arguments may be carried out for the general case as well.

Recall the basic Vapnik-Chervonenkis inequality [41], [39],

$$\mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left(L(f)-\widehat{L}_n(f)\right)>\epsilon\right]\le 4\mathbf{E}S_k(X_1^{2n})e^{-n\epsilon^2}, \tag{4}$$

where $S_k(X_1^n)$ is the *empirical shatter coefficient* of $\mathcal{F}_k$, that is, the number of different ways the n points $X_1,\ldots,X_n$ can be classified by elements of $\mathcal{F}_k$. It is easy to show that this inequality implies that the estimate

$$R_{n,k}=\widehat{L}_n(\widehat{f}_k)+\sqrt{\frac{\log\mathbf{E}S_k(X_1^{2n})+\log 4}{n}}$$

satisfies Assumption 1 with $m = n/2$. We need to estimate the quantity $\log\mathbf{E}S_k(X_1^{2n})$. The simplest way is to use the fact that $\mathbf{E}S_k(X_1^{2n}) \le (2en/V_k)^{V_k}$, where V_k is the VC dimension of $\mathcal{F}_k$. Substituting this into Theorem 1 gives

$$\begin{aligned}&\mathbf{E}L(f_n)-L^*\\&\le \min_k\left[\sqrt{\frac{V_k\log(2n)+\log 4}{n}}+\left(\inf_{f\in\mathcal{F}_k}L(f)-L^*\right)\right.\\&\qquad\left.+\sqrt{\frac{2\log k}{n}}\right]+\sqrt{\frac{1}{n}}.\end{aligned}\tag{5}$$

This is the type of distribution-free result we mentioned in the introduction. A more interesting result involves estimating $\mathbf{E}S_k(X_1^{2n})$ by $S_k(X_1^n)$.

Theorem 6 *Assume that the model selection algorithm of Section 2 is used with*

$$R_{n,k}=\widehat{L}_n(\widehat{f}_k)+\sqrt{\frac{12\log S_k(X_1^n)+\log 4}{n}}$$

and $m = n/80$. Then

$$\begin{aligned}&\mathbf{E}L(f_n)-L^*\\&\le \min_k\left[\sqrt{\frac{12\mathbf{E}\log S_k(X_1^n)+\log 4}{n}}\right.\\&\qquad\left.+\left(\inf_{f\in\mathcal{F}_k}L(f)-L^*\right)+8.95\sqrt{\frac{\log k}{n}}\right]+\frac{8.23}{\sqrt{n}}.\end{aligned}$$

The key ingredient of the proof is a concentration inequality from [10] for the *random* VC *entropy*, $\log_2 S_k(X_1^n)$.

Proof. We need to check the validity of Assumption 1. It is shown in [10] that $f(x_1,\ldots,x_n) = \log_2 S_k(x_1^n)$ satisfies the conditions of Theorem 4.

First note that $\mathbf{E}S_k(X_1^{2n}) \le \mathbf{E}^2 S_k(X_1^n)$, and therefore

$$\begin{aligned}&\log\mathbf{E}S_k(X_1^{2n})\\&\le 2\log\mathbf{E}S_k(X_1^n)\\&\le \frac{2}{\log 2}\mathbf{E}\log S_k(X_1^n)\\&< 3\mathbf{E}\log S_k(X_1^n)\end{aligned}$$

by the last inequality of Theorem 4. Therefore,

$$\begin{aligned}&\mathbf{P}\left[L(\widehat{f}_k)-\widehat{L}_n(\widehat{f}_k)>\epsilon+\sqrt{\frac{3\mathbf{E}\log S_k(X_1^n)+\log 4}{n}}\right]\\&\le \mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left(L(f)-\widehat{L}_n(f)\right)\right.\\&\qquad\left.>\epsilon+\sqrt{\frac{\log\mathbf{E}S_k(X_1^{2n})+\log 4}{n}}\right]\le e^{-n\epsilon^2},\end{aligned}$$

where we used the Vapnik-Chervonenkis inequality (4). It follows that

$$\begin{aligned}&\mathbf{P}\left[L(\widehat{f}_k)>R_{n,k}+\epsilon\right]\\&= \mathbf{P}\left[L(\widehat{f}_k)-\widehat{L}_n(\widehat{f}_k)\right.\\&\qquad\left.>\sqrt{\frac{12\log S_k(X_1^n)+\log 4}{n}}+\epsilon\right]\\&\le \mathbf{P}\left[L(\widehat{f}_k)-\widehat{L}_n(\widehat{f}_k)\right.\\&\qquad\left.>\frac{\epsilon}{4}+\sqrt{\frac{3\mathbf{E}\log S_k(X_1^n)+\log 4}{n}}\right]\\&\quad+\mathbf{P}\left[\sqrt{\frac{12\log S_k(X_1^n)+\log 4}{n}}+\frac{3\epsilon}{4}\right.\\&\qquad\left.<\sqrt{\frac{3\mathbf{E}\log S_k(X_1^n)+\log 4}{n}}\right]\\&\le e^{-n\epsilon^2/16}+\mathbf{P}\left[\sqrt{\frac{12\log S_k(X_1^n)+\log 4}{n}}+\frac{3\epsilon}{4}\right.\\&\qquad\left.<\sqrt{\frac{3\mathbf{E}\log S_k(X_1^n)+\log 4}{n}}\right].\end{aligned}$$

The last term may be bounded using Theorem 4 as follows:

$$\begin{aligned}&\mathbf{P}\left[\sqrt{\frac{12\log S_k(X_1^n)+\log 4}{n}}+\frac{3\epsilon}{4}\right.\\&\qquad\left.<\sqrt{\frac{3\mathbf{E}\log S_k(X_1^n)+\log 4}{n}}\right]\\&\le \mathbf{P}\left[\log S_k(X_1^n)\right.\\&\qquad\left.<\mathbf{E}\log S_k(X_1^n)-\frac{3}{4}\mathbf{E}\log S_k(X_1^n)-\frac{3}{64}n\epsilon^2\right]\end{aligned}$$

$$\leq \exp\left(-\frac{9}{32}\frac{\left(\mathbf{E}\log S_k(X_1^n)+\frac{n\epsilon^2}{16\log 2}\right)^2}{\mathbf{E}\log S_k(X_1^n)}\right)$$

$$\leq \exp\left(-\frac{9}{32}\frac{\left(\mathbf{E}\log S_k(X_1^n)+\frac{n\epsilon^2}{16\log 2}\right)^2}{\mathbf{E}\log S_k(X_1^n)+\frac{n\epsilon^2}{16\log 2}}\right)$$

$$\leq \exp\left(-\frac{9n\epsilon^2}{512\log 2}\right).$$

Summarizing, we have that

$$\begin{aligned}\mathbf{P}\left[L(\widehat{f}_k) > R_{n,k}+\epsilon\right] &\leq e^{-n\epsilon^2/16}+e^{-9n\epsilon^2/512\log 2}\\ &< 2e^{-n\epsilon^2/40}.\end{aligned}$$

Therefore, Assumption 1 is satisfied with $c = 2$ and $m = n/80$. Applying Theorem 1 finishes the proof. ∎

4.3 EFFECTIVE VC DIMENSION AND MARGIN

In practice it may be difficult to compute the value of the random shatter coefficients $S_k(X_1^n)$. An alternative way to assign complexities may be easily obtained by observing that $S_k(X_1^n) \leq (n+1)^{D_k}$, where D_k is the *empirical* VC dimension of class $\mathcal{F}_k$, that is, the VC dimension restricted to the points $X_1, \ldots, X_n$. Now it is immediate that the estimate

$$R_{n,k} = \widehat{L}_n(\widehat{f}_k) + \sqrt{\frac{12D_k\log(n+1)+\log 4}{n}},$$

satisfies Assumption 1 in the same way as the estimate of Theorem 6. (In fact, with a more careful analysis it is possible to get rid of the $\log n$ factor at the price of an increased constant.)

Unfortunately, computing D_k in general is still very difficult. A lot of effort has been devoted to obtain upper bounds for D_k which are simple to compute. These bounds are handy in our framework, since any upper bound may immediately be converted into a complexity penalty. In particular, the margins-based upper bounds on misclassification probability for neural networks [6], support vector machines [34, 7, 40, 13], and convex combinations of classifiers [32, 26] immediately give complexity penalties and, through Theorem 1, performance bounds.

We recall here some facts which are at the basis of the theory of *support vector machines*, see Bartlett and Shawe-Taylor [7], Cristianini and Shawe-Taylor [13], Vapnik [40] and the references therein.

A model class $\mathcal{F}$ is called a class of (generalized) linear classifiers if there exists a function $\psi : \mathcal{X} \to \mathcal{R}^p$ such that $\mathcal{F}$ is the class of linear classifiers in $\mathcal{R}^p$, that is, the class of all prediction rules of the form

$$f(x) = \begin{cases} 1 & \text{if } \psi(x)^T w \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $w \in \mathcal{R}^p$ is a weight vector satisfying $\|w\| = 1$.

Much of the theory of support vector machines builds on the fact that the "effective" VC dimension of those generalized linear classifiers for which the minimal distance of the correctly classified data points to the separating hyperplane is larger than a certain "margin" may be bounded, independently of the linear dimension p, by a function of the margin. If for some constant $\gamma > 0$, $(2Y_i - 1)\psi(X_i)^T w \geq \gamma$ then we say that the linear classifier *correctly classifies* X_i *with margin* γ. We recall the following result:

Lemma 1 (BARTLETT AND SHAWE-TAYLOR [7]). *Let f_n be an arbitrary (possibly data dependent) linear classifier of the form*

$$f_n(x) = \begin{cases} 1 & \text{if } \psi(x)^T w_n \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $w_n \in \mathcal{R}^p$ is a weight vector satisfying $\|w_n\| = 1$. Let $R, \gamma > 0$ be positive random variables and let $K \leq n$ be a positive integer valued random variable such that $\|\psi(X_i)\| \leq R$ for all $i = 1, \ldots, n$ and f_n correctly classifies all but K of the n data points X_i with margin γ, then for all $\delta > 0$,

$$\mathbf{P}\left[L(f_n) > \frac{K}{n} + 27.18\sqrt{\frac{1}{n}\left(\frac{R^2}{\gamma^2}(\log^2 n + 84) + \log\frac{4}{\delta}\right)}\,\right] \leq \delta.$$

Assume now that $\widehat{f}$ minimizes the empirical loss in a class $\mathcal{F}$ of generalized linear classifiers, such that it correctly classifies at least $n - K$ data points with margin γ and $\|\psi(X_i)\| \leq R$ for all $i = 1, \ldots, n$. Choosing $m = n\log 2/8$ and $\delta = 4e^{-2m\epsilon^2}$, an application of the lemma shows that if we take

$$R_n = \frac{K}{n} + 27.18\sqrt{\frac{1}{n}\left(\frac{R^2}{\gamma^2}(\log^2 n + 84)\right)},$$

then we obtain

$$\begin{aligned}&\mathbf{P}\left[L(\widehat{f}) > R_n + \epsilon\right]\\ &= \mathbf{P}\left[L(\widehat{f}) > \frac{K}{n} + 27.18\sqrt{\frac{1}{n}\left(\frac{R^2}{\gamma^2}(\log^2 n + 84)\right)} + \sqrt{\frac{1}{2m}\log\frac{4}{\delta}}\,\right]\\ &\leq \mathbf{P}\left[L(\widehat{f}) > \frac{K}{n} + 27.18\sqrt{\frac{1}{n}\left(\frac{R^2}{\gamma^2}(\log^2 n + 84) + \log\frac{4}{\delta}\right)}\,\right]\\ &\qquad \text{(using the inequality } \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}\text{)}\\ &\leq \delta = 4e^{-2m\epsilon^2}.\end{aligned}$$

This inequality shows that if all model classes $\mathcal{F}_k$ are classes of generalized linear classifiers and for all classes the error estimate $R_{n,k}$ is defined as above, then condition (1) is satisfied and Theorem 1 may be used. As a result, we obtain the following performance bound:

Theorem 7

$$\mathbf{E}L(f_n) - L^* \le \min_k \left[\mathbf{E}\left[\frac{K_k}{n} + 27.18\sqrt{\frac{1}{n}\left(\frac{R_k^2}{\gamma_k^2}(\log^2 n + 41)\right)} - \widehat{L}(\widehat{f}_k)\right] + \left(\inf_{f\in\mathcal{F}_k} L(f) - L^*\right) + 3.4\sqrt{\frac{\log k}{n}}\right] + \frac{3.72}{\sqrt{n}},$$

where K_k, γ_k, and R_k are the random variables K, γ, R defined above, corresponding to the class $\mathcal{F}_k$.

The importance of this result lies in the fact that it gives a computationally feasible way of assigning data-dependent penalties to linear classifiers. On the other hand, the estimates $R_{n,k}$ may be much inferior to the estimates studied in the previous section.

4.4 PENALIZATION BY MAXIMAL DISCREPANCY

In this section we propose an alternative way of computing the penalties with improved performance guarantees. The new penalties may be still difficult to compute efficiently, but there is a better chance to obtain good approximate quantities as they are defined as solutions of an optimization problem.

Assume, for simplicity, that n is even, divide the data into two equal halves, and define, for each predictor f, the empirical loss on the two parts by

$$\widehat{L}_n^{(1)}(f) = \frac{2}{n}\sum_{i=1}^{n/2} \ell(f(X_i), Y_i)$$

and

$$\widehat{L}_n^{(2)}(f) = \frac{2}{n}\sum_{i=n/2+1}^{n} \ell(f(X_i), Y_i).$$

Using the notation of Section 2, define the error estimate $R_{n,k}$ by

$$R_{n,k} = \widehat{L}_n(\widehat{f}_k) + \max_{f\in\mathcal{F}_k}\left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f)\right). \qquad (6)$$

If $\mathcal{Y} = \{0,1\}$ and the loss function is the 0-1 loss (i.e., $\ell(0,0) = \ell(1,1) = 0$ and $\ell(0,1) = \ell(1,0) = 1$) then the maximum discrepancy $\max_{f\in\mathcal{F}_k}\left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f)\right)$ may be computed using the following simple trick: first flip the labels of the first half of the data, thus obtaining the modified data set $D_n' = (X_1', Y_1'), \ldots, (X_n', Y_n')$ with $(X_i', Y_i') = (X_i, 1 - Y_i)$ for $i \le n/2$ and $(X_i', Y_i') = (X_i, Y_i)$ for $i > n/2$. Next find $f_k^- \in \mathcal{F}_k$ which minimizes the empirical loss based on D_n',

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\ell(f(X_i'), Y_i') \\
&= \frac{1}{2} - \frac{1}{n}\sum_{i=1}^{n/2}\ell(f(X_i), Y_i) + \frac{1}{n}\sum_{i=n/2+1}^{n}\ell(f(X_i), Y_i) \\
&= \frac{1 - \widehat{L}_n^{(1)}(f) + \widehat{L}_n^{(2)}(f)}{2}.
\end{aligned}$$

Clearly, the function f_k^- maximizes the discrepancy. Therefore, the same algorithm that is used to compute the empirical loss minimizer $\widehat{f}_k$ may be used to find f_k^- and compute the penalty based on maximum discrepancy. This is appealing: although empirical loss minimization is often computationally difficult, the same approximate optimization algorithm can be used for both finding prediction rules and estimating appropriate penalties. In particular, if the algorithm only approximately minimizes empirical loss over the class $\mathcal{F}_k$ because it minimizes over some proper subset of $\mathcal{F}_k$, the theorem is still applicable.

Vapnik *et al.* [43] considered a similar quantity for the case of pattern classification. Motivated by bounds (similar to (5)) on $\mathbf{E}L(f_n) - \widehat{L}_n(f)$, they defined an *effective* VC *dimension*, which is obtained by choosing a value of the VC dimension that gives the best fit of the bound to experimental estimates of $\mathbf{E}L(f_n) - \widehat{L}_n(f)$. They showed that for linear classifiers in a fixed dimension with a variety of probability distributions, the fit was good. This suggests a model selection strategy that estimates $\mathbf{E}L(f_n)$ using these bounds. The following theorem justifies a more direct approach (using discrepancy on the training data directly, rather than using discrepancy over a range of sample sizes to estimate effective VC dimension), and shows that an independent test sample is not necessary.

A similar estimate was considered in [45], although the error bound presented in [45, Theorem 3.4] can only be nontrivial when the maximum discrepancy is negative.

Theorem 8 *If the penalties are defined using the maximum-discrepancy error estimates (6), and $m = n/21$, then*

$$\begin{aligned}
&\mathbf{E}L(f_n) - L^* \\
&\le \min_k \left[\mathbf{E}\max_{f\in\mathcal{F}_k}\left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f)\right) + \left(\inf_{f\in\mathcal{F}_k} L(f) - L^*\right) + 4.59\sqrt{\frac{\log k}{n}}\right] + \frac{4.70}{\sqrt{n}}.
\end{aligned}$$

Proof. Once again, we check Assumption 1 and apply Theorem 1. Introduce the ghost sample $(X_1', Y_1'), \ldots, (X_n', Y_n')$, which is independent of the data and has the same distribution. Denote the empirical loss based on this sample by $L_n'(f) = \frac{1}{n}\sum_{i=1}^n \ell(f(X_i'), Y_i')$. The proof is based on the simple observation that for each k,

$$\begin{aligned}
&\mathbf{E}\max_{f\in\mathcal{F}_k}\left(L_n'(f) - \hat{L}_n(f)\right) \\
&= \frac{1}{n}\mathbf{E}\max_{f\in\mathcal{F}_k}\sum_{i=1}^{n}\left(\ell(f(X_i'), Y_i') - \ell(f(X_i), Y_i)\right) \\
&\le \frac{1}{n}\mathbf{E}\left(\max_{f\in\mathcal{F}_k}\sum_{i=1}^{n/2}\left(\ell(f(X_i'), Y_i') - \ell(f(X_i), Y_i)\right) + \max_{f\in\mathcal{F}_k}\sum_{i=n/2+1}^{n}\left(\ell(f(X_i'), Y_i') - \ell(f(X_i), Y_i)\right)\right) \\
&= \frac{2}{n}\mathbf{E}\max_{f\in\mathcal{F}_k}\sum_{i=1}^{n/2}\left(\ell(f(X_i'), Y_i') - \ell(f(X_i), Y_i)\right)
\end{aligned}$$

$$= \quad \mathbf{E} \max_{f \in \mathcal{F}_k} \left(\hat{L}_n^{(1)}(f) - \hat{L}_n^{(2)}(f) \right) . \tag{7}$$

McDiarmid's inequality (Theorem 3) implies

$$\mathbf{P} \left[\max_{f \in \mathcal{F}_k} \left(L'_n(f) - \widehat{L}_n(f) \right) > \mathbf{E} \max_{f \in \mathcal{F}_k} \left(L'_n(f) - \widehat{L}_n(f) \right) + \epsilon \right] \leq e^{-n\epsilon^2}, \tag{8}$$

$$\mathbf{P} \left[\max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) < \mathbf{E} \max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) - \epsilon \right] \leq e^{-n\epsilon^2/2} \tag{9}$$

and so for each k,

$$\begin{aligned}
&\mathbf{P} \left[L(\widehat{f}_k) > R_{n,k} + \epsilon \right] \\
&= \quad \mathbf{P} \left[L(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k) > \max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) + \epsilon \right] \\
&\leq \quad \mathbf{P} \left[L'_n(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k) > \max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) + \frac{7\epsilon}{9} \right] \\
&\qquad + \mathbf{P} \left[L(\widehat{f}_k) - L'_n(\widehat{f}_k) > \frac{2\epsilon}{9} \right] \\
&\leq \quad \mathbf{P} \left[L'_n(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k) > \max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) + \frac{7\epsilon}{9} \right] \\
&\qquad + e^{-8n\epsilon^2/81} \quad \text{(by Hoeffding)} \\
&\leq \quad \mathbf{P} \left[\max_{f \in \mathcal{F}_k} \left(L'_n(f) - \widehat{L}_n(f) \right) > \max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) + \frac{7\epsilon}{9} \right] \\
&\qquad + e^{-8n\epsilon^2/81} \\
&\leq \quad \mathbf{P} \left[\max_{f \in \mathcal{F}_k} \left(L'_n(f) - \widehat{L}_n(f) \right) > \mathbf{E} \max_{f \in \mathcal{F}_k} \left(L'_n(f) - \widehat{L}_n(f) \right) + \frac{\epsilon}{3} \right] \\
&\qquad + \mathbf{P} \left[\max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) < \mathbf{E} \max_{f \in \mathcal{F}_k} \left(\widehat{L}_n^{(1)}(f) - \widehat{L}_n^{(2)}(f) \right) - \frac{4\epsilon}{9} \right] \\
&\qquad + e^{-8n\epsilon^2/81} \quad \text{(where we used (7))} \\
&\leq \quad e^{-n\epsilon^2/9} + e^{-8n\epsilon^2/81} + e^{-8n\epsilon^2/81} \quad \text{(by (8) and (9))} \\
&< \quad 3e^{-8n\epsilon^2/81}.
\end{aligned}$$

Thus, Assumption 1 is satisfied with $m = n/21$ and $c = 3$ and the proof is finished. ∎

4.5 A RANDOMIZED COMPLEXITY ESTIMATOR

In this section we introduce an alternative way of estimating $\mathbf{E} \max_{f \in \mathcal{F}_k} \left(L(f) - \widehat{L}_n(f) \right)$ which may serve as an effective estimate of the complexity of a model class $\mathcal{F}$. The maximum discrepancy estimate of the previous section does this by splitting the data into two halves. Here we offer an alternative which allows us to derive improved performance bounds: we consider the expectation, over a random split of the data into two sets, of the maximal discrepancy.

Let $\sigma_1, \ldots, \sigma_n$ be a sequence of i.i.d. random variables such that $\mathbf{P}\{\sigma_i = 1\} = \mathbf{P}\{\sigma_i = -1\} = \frac{1}{2}$ and the σ_i's are independent of the data D_n. Introduce the quantity

$$M_{n,k} = \mathbf{E} \left[\sup_{f \in \mathcal{F}_k} \frac{2}{n} \sum_{i=1}^{n} \sigma_i \ell(f(X_i), Y_i) \middle| D_n \right] . \tag{10}$$

We use $M_{n,k}$ to measure the amount of overfitting in class $\mathcal{F}_k$. Note that $M_{n,k}$ is not known, but it may be computed with arbitrary precision by Monte-Carlo simulation. In the case of pattern classification, each computation in the integration involves minimizing empirical loss on a sample with randomly flipped labels. We offer two different ways of using these estimates for model selection. The first is based on Theorem 1 and the second, with a slight modification, on Theorem 2. We start with the simpler version:

Theorem 9 *Let $m = n/9$, and define the error estimates $R_{n,k} = \widehat{L}_n(\widehat{f}_k) + M_{n,k}$, and choose f_n by minimizing the penalized error estimates*

$$\tilde{L}_n(\widehat{f}_k) = \widehat{L}_n(\widehat{f}_k) + C_n(k) = R_{n,k} + \sqrt{\frac{\log k}{m}}.$$

then

$$\begin{aligned}
&\mathbf{E}L(f_n) - L^* \\
&\leq \quad \min_k \left[\mathbf{E}M_{n,k} + \left(\inf_{f \in \mathcal{F}_k} L(f) - L^* \right) + 3\sqrt{\frac{\log k}{n}} \right] \\
&\qquad + \frac{2.77}{\sqrt{n}} .
\end{aligned}$$

Proof. Introduce a ghost sample as in the proof of Theorem 8, and recall that by a symmetrization trick of Giné and Zinn [18],

$$\begin{aligned}
&\mathbf{E} \left[\sup_{f \in \mathcal{F}_k} (L(f) - L_n(f)) \right] \\
&= \quad \mathbf{E} \left[\sup_{f \in \mathcal{F}_k} \mathbf{E} \left[L'_n(f) - L_n(f) \middle| D_n \right] \right] \\
&\leq \quad \mathbf{E} \left[\sup_{f \in \mathcal{F}_k} (L'_n(f) - L_n(f)) \right] \\
&= \quad \frac{1}{n} \mathbf{E} \left[\sup_{f \in \mathcal{F}_k} \sum_{i=1}^{n} \sigma_i (\ell(f(X'_i), Y'_i) - \ell(f(X_i), Y_i)) \right] \\
&\leq \quad \frac{2}{n} \mathbf{E} \left[\sup_{f \in \mathcal{F}_k} \sum_{i=1}^{n} \sigma_i \ell(f(X_i), Y_i) \right] \\
&= \quad \mathbf{E}M_{n,k} .
\end{aligned} \tag{11}$$

The rest of the proof of Assumption 1 follows easily from concentration inequalities: for each k,

$$\begin{aligned}
&\mathbf{P}\left[L(\widehat{f}_k) > R_{n,k} + \epsilon\right] \\
&= \mathbf{P}\left[L(\widehat{f}_k) - \widehat{L}_n(\widehat{f}_k) > M_{n,k} + \epsilon\right] \\
&\leq \mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left(L(f) - \widehat{L}_n(f)\right) > M_{n,k} + \epsilon\right] \\
&\leq \mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left(L(f) - \widehat{L}_n(f)\right)\right. \\
&\qquad \left. > \mathbf{E}\sup_{f\in\mathcal{F}_k}\left(L(f) - \widehat{L}_n(f)\right) + \frac{\epsilon}{3}\right] \\
&\qquad + \mathbf{P}\left[\mathbf{E}\sup_{f\in\mathcal{F}_k}\left(L(f) - \widehat{L}_n(f)\right) > M_{n,k} + \frac{2\epsilon}{3}\right] \\
&\leq \mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left(L(f) - \widehat{L}_n(f)\right)\right. \\
&\qquad \left. > \mathbf{E}\sup_{f\in\mathcal{F}_k}\left(L(f) - \widehat{L}_n(f)\right) + \frac{\epsilon}{3}\right] \\
&\qquad + \mathbf{P}\left[\mathbf{E}M_{n,k} > M_{n,k} + \frac{2\epsilon}{3}\right] \qquad \text{(by (11))} \\
&\leq 2e^{-2n\epsilon^2/9},
\end{aligned}$$

where at the last step we used McDiarmid's inequality. (It is easy to verify that $M_{n,k}$ and $\sup\left(L(f) - \widehat{L}_n(f)\right)$ satisfy the condition of Theorem 3 with $c_i = 2/n$ and $c_i = 1/n$, respectively.) Thus, Assumption 1 holds with $c = 2$ and $m = n/9$. Theorem 1 implies the result. ∎

The following theorem shows that we can get rid of the square root signs at the expense of slightly increasing the complexity penalty. This improvement is important when the class $\mathcal{F}_k$ has $\mathbf{E}M_{n,k}$ much smaller than $n^{-1/2}$. The key difference in the proof is the use of the refined concentration inequalities from [10] instead of McDiarmid's inequality.

Introduce the modified error estimate

$$\overline{R}_{n,k} = \widehat{L}_n(\widehat{f}_k) + \overline{M}_{n,k},$$

where

$$\overline{M}_{n,k} = \mathbf{E}\left[\sup_{f\in\mathcal{F}_k}\frac{256}{n}\left|\sum_{i=1}^{n}\sigma_i\ell(f(X_i), Y_i)\right|\,\middle|\, D_n\right]. \quad (12)$$

Note that $\overline{M}_{n,k}$ is basically a constant factor times $M_{n,k}$. (The constants have not been optimized.)

Theorem 10 *Let $m = n/4096$, and choose $\overline{f}_n$ by minimizing the penalized error estimates*

$$\overline{L}_n(\widehat{f}_k) = \widehat{L}_n(\widehat{f}_k) + \overline{C}_n(k) = \overline{R}_{n,k} + \frac{2\log k}{m}.$$

Then

$$\begin{aligned}
&\mathbf{E}L(f_n) - L^* \\
&\leq \min_k\left[\mathbf{E}\overline{M}_{n,k} + \left(\inf_{f\in\mathcal{F}_k} L(f) - L^*\right) + \frac{8192\log k}{n}\right] \\
&\qquad + \frac{13096}{n}.
\end{aligned}$$

In the proof we use some auxiliary results. The first is called Khinchine's inequality:

Lemma 2 (Szarek [36]) *Suppose $\sigma_1, \ldots, \sigma_n$ are symmetric i.i.d. sign variables, and let $a_1, \ldots, a_n$ be real numbers. Then*

$$\mathbf{E}\left|\sum_{i=1}^{n}\sigma_i a_i\right| \geq \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{n}a_i^2}\,.$$

The next lemma concerns a simple property of $[0,1]$-valued random variables:

Lemma 3 *For n i.i.d. random variables $X_i \in [0,1]$ with $\mathbf{E}X_i = p \geq 4/(n+4)$, the sum $Z = \sum_{i=1}^{n} X_i$ satisfies*

$$\mathbf{E}\sqrt{Z} \geq \frac{\sqrt{np}}{2}\,.$$

Proof.

$$\begin{aligned}
\mathbf{E}\sqrt{Z} &= \mathbf{E}\left(\sqrt{Z} - \sqrt{np}\right) + \sqrt{np} \\
&\geq \sqrt{np} - \mathbf{E}\left|\sqrt{Z} - \sqrt{np}\right| \\
&\geq \sqrt{np} - \frac{\mathbf{E}\left|Z - np\right|}{\sqrt{np}} \\
&\qquad \text{(using } |\sqrt{a} - \sqrt{b}| \leq |a-b|/\sqrt{a}\text{)} \\
&\geq \sqrt{np} - \frac{\sqrt{\mathrm{Var}(Z)}}{\sqrt{np}} \quad \text{(by Cauchy-Schwarz)} \\
&\geq \sqrt{np} - \frac{\sqrt{np(1-p)}}{\sqrt{np}} \\
&= \sqrt{np} - \sqrt{1-p},
\end{aligned}$$

and the result follows. ∎

Next we need a classical symmetrization inequality from empirical process theory:

Lemma 4 (Giné and Zinn [18]). *Let $\mathcal{F}$ be a class of real-valued functions defined on a set A, let $X_1, \ldots, X_n$ be i.i.d. random variables taking their value in A, and let $\sigma_1, \ldots, \sigma_n$ be symmetric i.i.d. sign variables. If*

$$\Sigma^2 = \sup_{f\in\mathcal{F}}\mathrm{Var} f(X_1),$$

then for all $t > \Sigma\sqrt{8n}$,

$$\begin{aligned}
&\mathbf{P}\left[\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}(f(X_i) - \mathbf{E}f(X_i))\right| > t\right] \\
&\leq 4\mathbf{P}\left[\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}\sigma_i f(X_i)\right| > \frac{t}{4}\right].
\end{aligned}$$

Finally, we show that the penalty term is sharply concentrated around its mean.

Lemma 5 *Consider the following function $Q_{n,k} : (\mathcal{X} \times \mathcal{Y})^n \to [0,n]$:*

$$Q_{n,k} \stackrel{\text{def}}{=} \mathbf{E}\left[\sup_{f\in\mathcal{F}_k}\left|\sum_{i=1}^n \sigma_i \ell(f(x_i),y_i)\right|\right] = \frac{n}{256}\overline{M}_{n,k} .$$

Then $Q_{n,k}$ satisfies the conditions of Theorem 4.

Proof. Clearly, $Q_{n,k}$ is nonnegative. To check condition (2) of Theorem 4, for every $i \le n$ introduce

$$Q_{n,k}^{(i)} = \mathbf{E}\left[\max_{f\in\mathcal{F}_k}\left|\sum_{j\neq i}\sigma_j\ell(f(x_j),y_j)\right|\right] . \tag{13}$$

Clearly,

$$\begin{aligned} & Q_{n,k}^{(i)} && (14)\\ &= \mathbf{E}\left[\max_{f\in\mathcal{F}_k}\left|\sum_{j\neq i}\sigma_j\ell(f(x_j),y_j)+\mathbf{E}\sigma_i\ell(f(x_i),y_i)\right|\right] \\ &\le \overline{M}_{n,k} && (15)\end{aligned}$$

and $Q_{n,k} - Q_{n,k}^{(i)} \le 1$. Finally, to check condition (3) of Theorem 4, for each realization of $(\sigma_j)_{j\le n}$, let f_σ be such that

$$\max_{f\in\mathcal{F}_k}\left|\sum_{j=1}^n\sigma_j\ell(f(x_j),y_j)\right| = \alpha\sum_{j=1}^n\sigma_j\ell(f_\sigma(x_j),y_j)$$

where $\alpha=\pm1$. Then

$$\begin{aligned} &\sum_{i=1}^n\left(Q_{n,k}-Q_{n,k}^{(i)}\right)\\ &\quad= \mathbf{E}\sum_{i=1}^n\left[\max_{f\in\mathcal{F}_k}\left|\sum_j\sigma_j\ell(f(x_j),y_j)\right| \right.\\ &\qquad\qquad \left. -\max_{f\in\mathcal{F}_k}\left|\sum_{j\neq i}\sigma_j\ell(f(x_j),y_j)\right|\right]\\ &\quad\le \mathbf{E}\sum_{i=1}^n\left[\alpha\left(\sum_{j=1}^n\sigma_j\ell(f_\sigma(x_j),y_j)\right)\right.\\ &\qquad\qquad\left. -\alpha\left(\sum_{j\neq i}\sigma_j\ell(f_\sigma(x_j),y_j)\right)\right]\\ &\quad= \mathbf{E}\sum_{i=1}^n\alpha\sigma_i\ell(f_\sigma(x_i),y_i)\\ &\quad= Q_{n,k} .\end{aligned}$$

∎

Proof of Theorem 10. We check Assumption 2 and apply Theorem 2. We have

$$\begin{aligned} &\mathbf{P}\left[L(\widehat{f}_k) > \overline{R}_{n,k}+\epsilon\right]\\ &\le \mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left|L(f)-\widehat{L}_n(f)\right| > \overline{M}_{n,k}+\epsilon\right]\\ &\le \mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left|L(f)-\widehat{L}_n(f)\right| > \frac{1}{2}\mathbf{E}\overline{M}_{n,k}+2\epsilon/3\right]\\ &\qquad + \mathbf{P}\left[\frac{1}{2}\mathbf{E}\overline{M}_{n,k} > \overline{M}_{n,k}+\epsilon/3\right]\\ &\stackrel{\text{def}}{=} I + II.\end{aligned}$$

To bound I, we note that by Lemma 4,

$$I \le 4\mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\frac{1}{n}\left|\sum_{i=1}^n\sigma_i\ell(f(X_i),Y_i)\right| > \frac{1}{8}\mathbf{E}\overline{M}_{n,k}+\frac{\epsilon}{6}\right] \tag{16}$$

whenever

$$\frac{1}{2}\mathbf{E}\overline{M}_{n,k}+\frac{2\epsilon}{3} > \sqrt{\frac{8}{n}}\sup_{f\in\mathcal{F}_k}\sqrt{\mathrm{Var}\ell\left(f(X_i),Y_i\right)}.$$

But the condition is satisfied for all $\epsilon > 3\sqrt{8}/n$, since if $\sup_{f\in\mathcal{F}_k}\mathrm{Var}\ell\left(f(X_i),Y_i\right) \le 4/n$, then

$$2\epsilon/3 \ge \sqrt{32}/n \ge \sqrt{\frac{8}{n}}\sup_{f\in\mathcal{F}_k}\sqrt{\mathrm{Var}\ell\left(f(X_i),Y_i\right)}.$$

On the other hand, if $\sup_{f\in\mathcal{F}_k}\mathrm{Var}\ell\left(f(X_i),Y_i\right) > 4/n$, then $\mathbf{E}\ell\left(f(X_i),Y_i\right)^2 > 4/n$, and so

$$\begin{aligned} &\frac{1}{2}\mathbf{E}\overline{M}_{n,k}\\ &= \mathbf{E}\left[\sup_{f\in\mathcal{F}_k}\frac{128}{n}\left|\sum_{i=1}^n\sigma_i\ell(f(X_i),Y_i)\right|\right]\\ &\ge \sup_{f\in\mathcal{F}_k}\mathbf{E}\left[\frac{128}{n}\left|\sum_{i=1}^n\sigma_i\ell(f(X_i),Y_i)\right|\right]\\ &= \frac{128}{n}\sup_{f\in\mathcal{F}_k}\mathbf{E}\mathbf{E}\left[\left|\sum_{i=1}^n\sigma_i\ell(f(X_i),Y_i)\right|\middle| D_n\right]\\ &\ge \frac{128}{n}\sup_{f\in\mathcal{F}_k}\mathbf{E}\left[\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^n\ell\left(f(X_i),Y_i\right)^2}\right]\\ &\qquad\text{(by Lemma 2)}\\ &\ge \frac{64}{\sqrt{2n}}\sup_{f\in\mathcal{F}_k}\sqrt{\mathbf{E}\ell\left(f(X_i),Y_i\right)^2} \qquad\text{(by Lemma 3)}\\ &> \sqrt{\frac{8}{n}}\sup_{f\in\mathcal{F}_k}\sqrt{\mathrm{Var}\ell\left(f(X_i),Y_i\right)}.\end{aligned}$$

Thus, we need to obtain a suitable upper bound for the probability on the right-hand side of (16). To this end, write

$$\begin{aligned}
&4\mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\frac{1}{n}\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right| > \frac{1}{8}\mathbf{E}\overline{M}_{n,k}+\frac{\epsilon}{6}\right] \\
&\leq\; 4\mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\frac{1}{n}\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right| \right. \\
&\qquad\qquad \left. > \frac{1}{16}\overline{M}_{n,k}+\frac{\epsilon}{12}\right] \\
&\qquad + 4\mathbf{P}\left[\frac{1}{16}\overline{M}_{n,k} > \frac{1}{8}\mathbf{E}\overline{M}_{n,k}+\frac{\epsilon}{12}\right] \\
&\stackrel{\text{def}}{=}\; III + IV\,.
\end{aligned}$$

We bound III by applying Theorem 5 to the random variable

$$\sup_{f\in\mathcal{F}_k}\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right|$$

conditionally, keeping D_n fixed. This function is easily seen to satisfy the conditions of Theorem 5, and therefore we obtain

$$\begin{aligned}
&III \\
&=\; 4\mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\frac{1}{n}\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right| \right. \\
&\qquad \left. > \mathbf{E}\left[\sup_{f\in\mathcal{F}_k}\frac{16}{n}\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right|\,\middle|\, D_n\right]+\frac{\epsilon}{12}\right] \\
&=\; 4\mathbf{P}\left[\sup_{f\in\mathcal{F}_k}\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right| \right. \\
&\qquad \left. > \mathbf{E}\left[\sup_{f\in\mathcal{F}_k}16\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right|\,\middle|\, D_n\right]+\frac{n\epsilon}{12}\right] \\
&\leq\; 4\exp\left(-\frac{n\epsilon}{24}\right. \\
&\quad \left.\times\log\left(\frac{\mathbf{E}\left[\sup_{f\in\mathcal{F}_k}16\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right|\,\middle|\, D_n\right]+\frac{n\epsilon}{12}}{\mathbf{E}\left[\sup_{f\in\mathcal{F}_k}8\left|\sum_{i=1}^n \sigma_i \ell(f(X_i),Y_i)\right|\,\middle|\, D_n\right]+4}\right)\right) \\
&\leq\; 4\exp\left(-\frac{n\epsilon\log 2}{24}\right)
\end{aligned}$$

whenever $\epsilon \geq 96/n$. Finally, we need to bound the probabilities II and IV. But this may be done by a straightforward application of Lemma 5 and Theorem 4. We obtain

$$\begin{aligned}
&II+IV \\
&=\; \mathbf{P}\left[\frac{1}{2}\mathbf{E}\overline{M}_{n,k} > \overline{M}_{n,k}+\epsilon/3\right] \\
&\qquad + 4\mathbf{P}\left[\frac{1}{16}\overline{M}_{n,k} > \frac{1}{8}\mathbf{E}\overline{M}_{n,k}+\frac{\epsilon}{12}\right] \\
&=\; \mathbf{P}\left[\frac{n}{256}\overline{M}_{n,k}\right. \\
&\qquad \left. < \frac{n}{256}\mathbf{E}\overline{M}_{n,k}-\frac{n}{512}\mathbf{E}\overline{M}_{n,k}-\frac{n\epsilon}{3\cdot 256}\right] \\
&\qquad + 4\mathbf{P}\left[\frac{n}{256}\overline{M}_{n,k}\right. \\
&\qquad \left. > \frac{n}{256}\mathbf{E}\overline{M}_{n,k}+\frac{n}{256}\mathbf{E}\overline{M}_{n,k}+\frac{n\epsilon}{16\cdot 12}\right] \\
&\leq\; 5\exp\left(-\frac{\left(\frac{n}{512}\mathbf{E}\overline{M}_{n,k}+\frac{n\epsilon}{3\cdot 256}\right)^2}{\frac{16}{3}\left(\frac{n}{512}\mathbf{E}\overline{M}_{n,k}+\frac{n\epsilon}{3\cdot 256}\right)}\right) \\
&\leq\; 5e^{-n\epsilon/4096}\,.
\end{aligned}$$

Collecting bounds, we obtain that for all $\epsilon \geq 96/n$,

$$\mathbf{P}\left[L(\widehat{f}_k) > \overline{R}_{n,k}+\epsilon\right] \leq 9e^{-n\epsilon/4096}\,.$$

It is easy to modify the proof of Theorem 2 to accommodate this restriction for ϵ (provided $96/n \leq \log(2c)/m$), and straightforward calculation yields the result. ∎

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