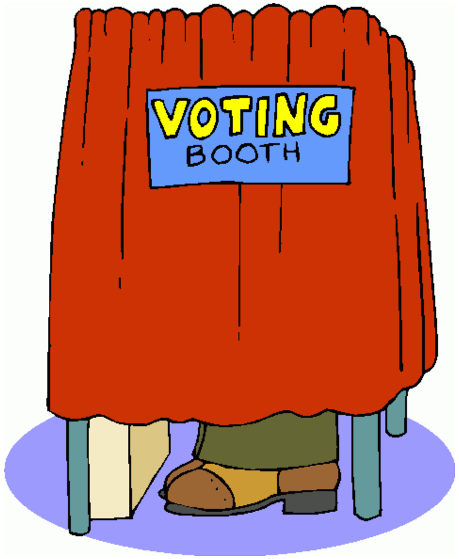
Voting Paradoxes

Noga Alon, Tel Aviv U. and Microsoft, Israel



COLT, June 2010

The Condorcet Paradox (1785):

The majority may prefer A to B, B to C and C to A.

Indeed, if the preferences of 3 voters are:

A>B>C

B>C>A

C>A>B

then

2/3 prefer A to B

2/3 prefer B to C

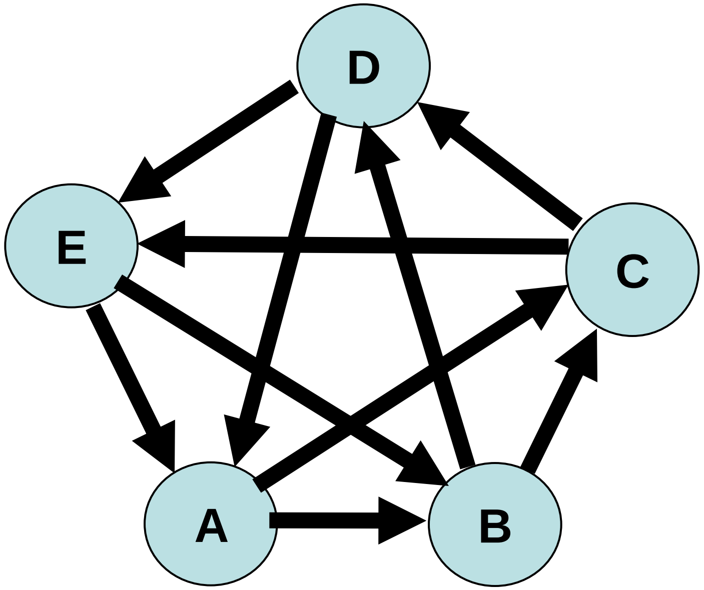
2/3 prefer C to A

The moral:

The majority preferences may be irrational

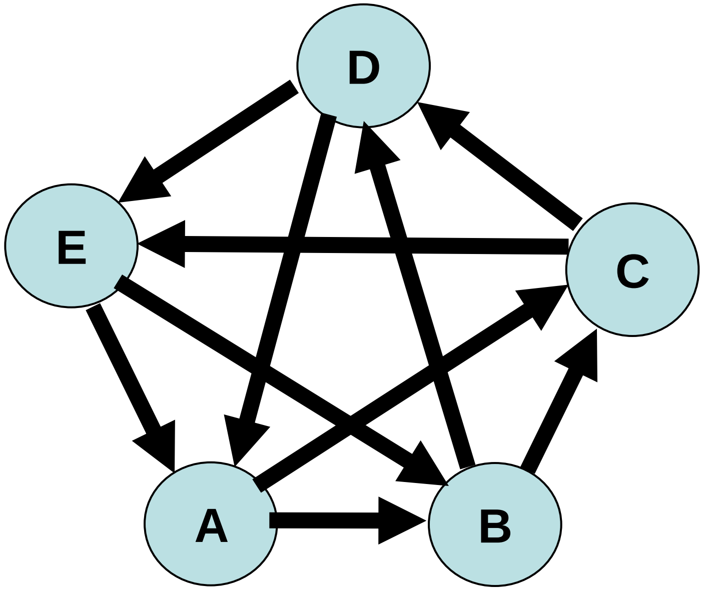
marquis de **Condorcet**

McGarvey (1953): The majority may exhibit any pattern of pairwise preferences



A>B>C>D>E
C>E>B>D>A
D>E>A>B>C

Def: A **tournament** is an oriented complete graph



Def: It is a **2k-1 majority tournament** if there are 2k-1 linear orders on the vertices, and (i,j) is a directed edge iff i precedes j in at least k of them.

McGarvey (53): Every tournament on n vertices is a 2k-1 majority tournament for $k \le O(n^2)$.

Stearns (59): $k \le O(n)$ orderes suffice

Erdős-Moser (64): $k \le O(n/ \log n)$ orderes suffice (that's tight)

Malla (99), A (02): Most tournaments on n vertices cannot be realized as majority tournaments with a gap of more than $c/ n^{1/2}$ in each edge.

The moral:

The majority preferences may be chaotic

Voting schemes

n voters, k candidates

Each voter in the group ranks all candidates (linearly), and the scheme provides the group's linear ranking of the candidates

Axiom 1 (unanimity): If all voters rank A above B, then so does the resulting order

Axiom 2 (independence of irrelevant alternatives): The group's relative ranking of any pair of candidates is determined by the voters relative ranking of this pair.


IN THAT CASE, I'LL HAVE A CHICKEN


SORRY, WE ALSO HAVE A FISH

Arrow (1951):

If k ≥3, the only scheme that satisfies axiom 1 and axiom 2 is dictatorship, that is, the group's ranking is determined by that of one voter !

The moral:

The only ``reasonable" voting scheme is dictatorship

Leader Election

n voters, k candidates
Each voter ranks all candidates linearly. The winner (=leader) is determined by these orderings following a known rule

Axiom 1: The rule is not dictatorship, that is, no single voter can choose the leader by himself

Axiom 2: Any candidate can win under the rule, with some profile of the voters' preferences.

Gibbard (1973), Satterthwaite (1975):

If $k \geq 3$, any such scheme can be manipulated, that is, there are cases in which a voter who knows the preferences of the other voters and knows the rule has an incentive to vote untruthfully

The moral:

Any reasonable leader election scheme can be manipulated

Back to the majority rule

Mossel, O'Donnell and Oleszkiewicz (05): Majority is the stablest balanced binary function with negligible influences

That is: if f maps $\{-1,1\}^n$ to {-1,1}, its expectation is 0, and each input bit has little influence on the outcome, then flipping each input bit randomly changes the outcome with probability at least that in which this happens for the majority.

Fellowships

A committee of size 2k-1 has to select r winners among n candidates.
Each committee member (=voter) provides a linear order of the candidates, and the scheme chooses r winners.

Axiom: For any profile of preferences, there is no non-winner A so that for every winner B, most of the committee members rank A over B

Remark: The example of Condorcet shows that this is impossible for 2k-1=3, r=1.

Alon,Brightwell,Kierstead,Kostochka,Winkler:

For 2k-1=3, if $r \le 2$ there is no such scheme,
$r \ge 3$ suffices

For larger k, if $r \le \frac{1}{3} k / \log k$ there is no such scheme,
$r \ge 80\, k \log k$ suffices.

In other words: every 2k-1 majority tournament has a dominating set of size at most $O(k \log k)$
[and there are examples with no such set of size $o(k/ \log k)$].

Sketch of proof:

For a tournament T=(V,E), let H(T) be the hypergraph on V whose edges are all sets $i \cup \{ j : (j,i) \in E \}$.

A cover of H(T) is a set of vertices hitting all edges.

Our objective is to show that if T is a 2k-1 majority tournament, then H(T) has a cover of size O(k log k).

A **fractional cover** of a hypergraph H is an assignment of weights to the vertices so that the weight of each edge is a least 1.

Fact 1: For any tournament T, the hypergraph H(T) has a fractional cover of total weight at most 2.

This is proved by applying Von-Newmann minimax theorem to the two-player zero-sum game in which each player selects a vertex of T, and the player with the winning vertex gets 1 $

Theorem [Haussler and Welzl (87), following Vapnik and Chervonenkis (71)]: If the VC-dimension of a hypergraph is at most d, and it has a fractional cover of weight t, then it has a cover of size at most O(d t log t).

Note: For H=(V,E), **VC(H)** is the maximum cardinality of a subset A of V so that every subset B of A satisfies B=e ∩ A for some e є E.

Fact 2: It T is a 2k-1 majority tournament, then the VC-dimension of H(T) is at most O(k log k).

This shows that r=O(k log k) winners suffice for a committee of 2k-1 members.

Examples showing that sometimes r = Ω (k log k) winners do not suffice are constructed by a probabilistic argument.

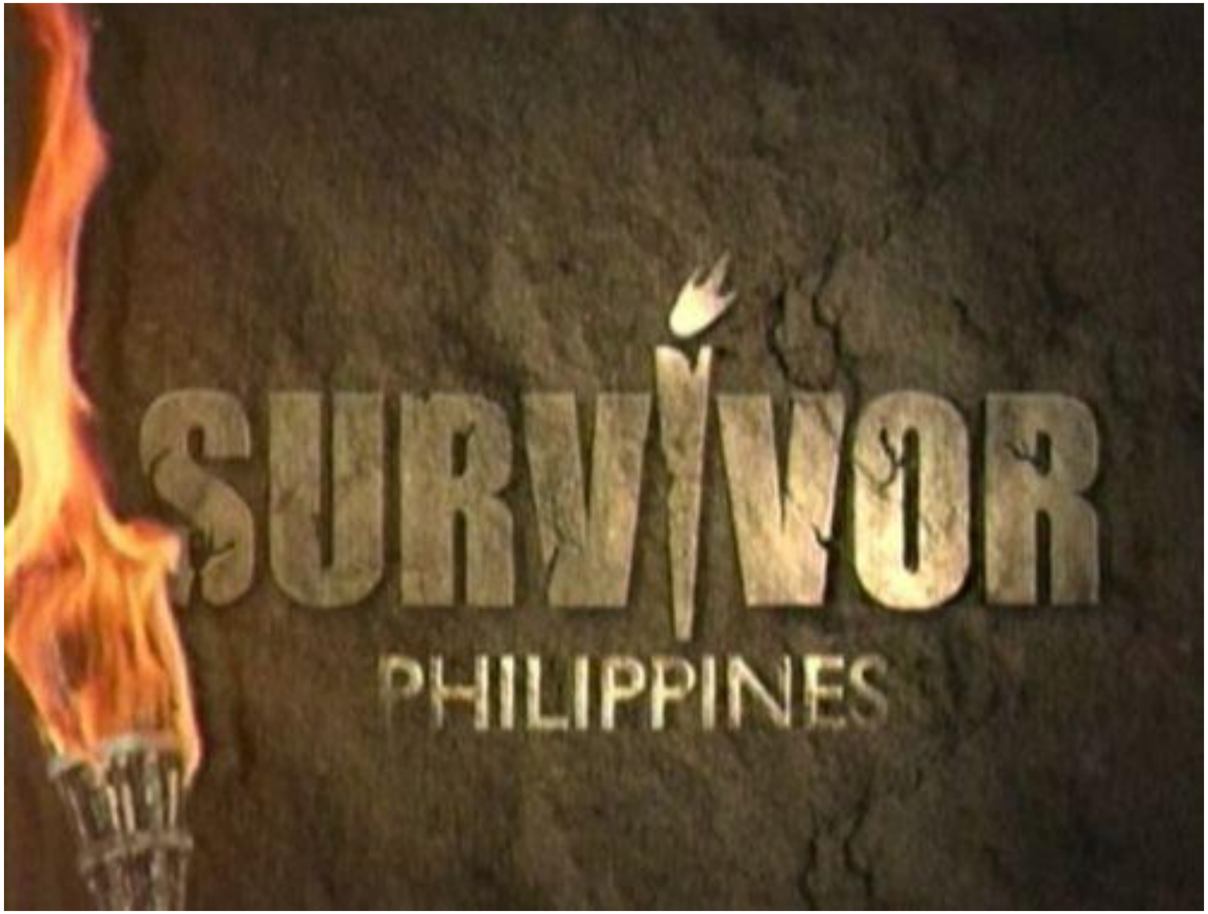
□

Open: What's the smallest possible r that suffices for a committee of 2k-1 members ?

The moral:

Bigger committees require bigger budget

Reality Games



In a variant of the TV show ``Survivor" each tribe member can recommend at most one other trusted member

The mechanism selects a member to be eliminated in the tribal council, based on these recommendations

Axiom: If there is a unique tribe member that received positive recommendations, then this member cannot be the eliminated one.

Alon, Fischer, Procaccia, Tennenholtz (2010):

No such scheme can be strategy-proof, that is, there must be a scenario in which a member, knowing the scheme and the recommendations of all others, can gain (=avoid being eliminated) by mis-reporting his recommendation.

Proof:

- Denote the tribe members by 0,1,..,n, and assume that when no positive votes are given, 0 is the one being eliminated.

- Consider the 2^n scenarios in which 0 does not vote, and each i between 1 and n either votes for 0 or for nobody.

- By the axiom, 0 is being eliminated only in one such scenario (when nobody recommends him).

- **By strategy-proofness, if i > 0 is being eliminated in some scenario, he is also the one to be eliminated when i changes his vote**

  Therefore, the total number of scenarios in which i is being eliminated is even.

- **But this is impossible, as the total number of scenarios considered is even.** □

The moral:

Cheating is inherent in reality games

(unless one uses randomization)

Summary (informal): we have seen

Condorcet (1785): The majority may be **irrational**

McGarvey (1953): The majority may be **chaotic**

Arrow (1951): The only reasonable voting scheme is **dictatorship**

GS (1973,75): Any reasonable leader election game can be **manipulated**

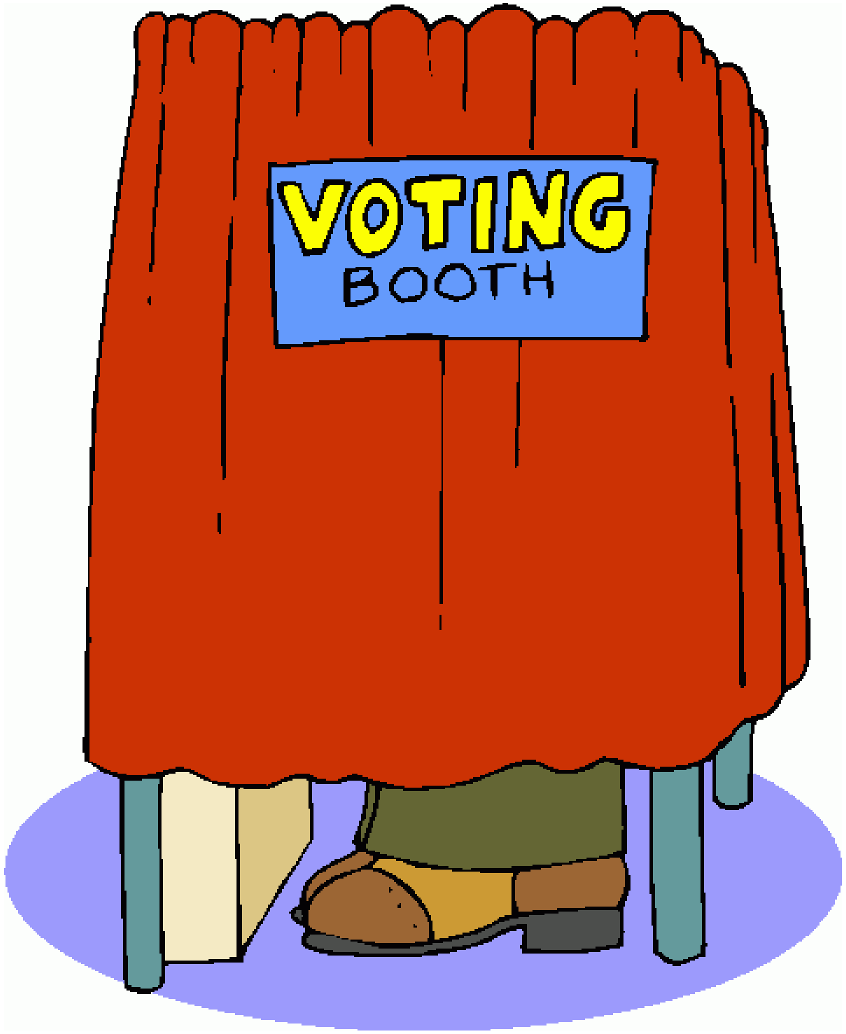
ABKKW: Bigger committees require **bigger budget**

AFPT: **Cheating** is inherent in reality games

Is the theory of Social Choice relevant to real life ?

Condorcet (1775):

``Rejecting theory as useless in order to work on everyday things is like proposing to cut the roots of a tree because they do not carry fruit"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